SRF Sypnosis

Xiao Suyang

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The aim of this work is to apply a variational quantum algorithm to price a European option modelled by the Black Scholes Equation with stochastic volatility.

For an European Option with Volatility $\sqrt{y_t}$, and an underlying stock with price x, we have two stochastic processes:

$$dx_t = \mu x_t dt + \sqrt{y_t} x_t dW_1$$
$$dy_t = aydt + bydW_2$$

Where the correlation between the two Wiener process dW_1 and dW_2 is given by ρ. Let $V(x, y, t)$ be the value of the option on the underlying x at time t. Then, using portfolio replication with Ito's lemma, coupled with non-arbitrage arguments gives the following PDE:

$$\frac{\partial u}{\partial t} + \left(rx\frac{\partial}{\partial x} + \frac{1}{2}x^2 y \frac{\partial^2}{\partial x^2} + b\rho x y^{\frac{3}{2}} \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} b^2 y^2 \frac{\partial^2}{\partial y^2} + ay\frac{\partial}{\partial y} - r \right) u = 0$$

To find the solution of the PDE using VarQITE, we can express this in a Schrodinger-type equation. First, we write

$$\frac{\partial u}{\partial t} = \mathfrak{G} u$$

With the infinitesimal generator

$$\mathfrak{G} = -\left(rx\frac{\partial}{\partial x} + \frac{1}{2}x^2 y \frac{\partial^2}{\partial x^2} + b\rho x y^{\frac{3}{2}} \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} b^2 y^2 \frac{\partial^2}{\partial y^2} + ay\frac{\partial}{\partial y} - r \right)$$

We consider the simplist case where a, b, r have no dependence on x, y, t and treat them as constants. For a numerical experiment, we set $a = b = r = 1$ where $\mathfrak{G}$ becomes

$$\mathfrak{G} = -\left(x\frac{\partial}{\partial x} + \frac{1}{2}x^2 y \frac{\partial^2}{\partial x^2} + \rho x y^{\frac{3}{2}} \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} y^2 \frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - 1 \right)$$

The first step is to set $\tau = T - t$ since we are evolving backwards at the final time T. We then apply a wick rotation $\epsilon = -i\tau$. With the substitution, we get

$$-i\frac{\partial u}{\partial \epsilon} = \widetilde{\mathfrak{G}} u$$

We want to simulate this evolution in an equivalent quantum system, with

$$-i\frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$

where

$$\hat{H} = \widetilde{\mathfrak{G}} = \left(x\frac{\partial}{\partial x} + \frac{1}{2}x^2 y\frac{\partial^2}{\partial x^2} + \rho x y^{\frac{3}{2}}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - \mathbb{I} \right)$$

Since $\hat{H}$ is not Hermitian in this case, and also that Along the imaginary axis, the corresponding evolution operator $\exp \hat{H}\tau$ is not unitary, we instead use a variational method to evolve a trial wavefunction $|\phi(\boldsymbol{\theta})\rangle$, parameterised by $\boldsymbol{\theta}$ that approximates $|\psi\rangle$.

For the ansatz $|\phi(\boldsymbol{\theta})\rangle$, we initialise it by finding the parameter θ_0 that brings this trial wavefunction as close to the original wavefunction we are starting with, where

$$\boldsymbol{\theta_0} = \underset{\theta \in \mathbf{R}^n}{\arg\min} \, \| |\phi(\boldsymbol{\theta})\rangle - |\psi(0)\rangle \|$$

This process is done classically, via some variant of gradient descent. The ansatz $|\phi(\boldsymbol{\theta})\rangle$ is prepared from the initial condition of the target function at $t = 0$, i.e. $u(x, y, 0)$. We simply match the amplitude of $|\phi(\boldsymbol{\theta})\rangle$ with $u(x, y, 0)$ on a grid.

Next, after decomposing $\hat{H}$ in to $\sum \lambda_i h_i$, a sum of unitary operators, we apply McLachlan's Variational principle to evolve $\boldsymbol{\theta}$ in imaginary time. This is equivalent to solving the following system of ODEs, where

$$A(t)\boldsymbol{\theta(t)}' = C(t)$$

With the matrix $A(t)$, vector $C(t)$ defined as the following:

$$A_{i,j}(t) = \Re\left(\frac{\partial \langle \phi(\boldsymbol{\theta}(t))|}{\partial \theta_i} \frac{\partial |\phi(\boldsymbol{\theta}(t))\rangle}{\partial \theta_j} \right) \tag{1}$$

$$C_i(t) = \Re\left(-\sum_i \lambda_i \frac{\partial \langle \phi(\boldsymbol{\theta}(t))|}{\partial \theta_i} h_i |\phi(\boldsymbol{\theta}(t))\rangle \right) \tag{2}$$

We can then evolve the parameter $\boldsymbol{\theta}$ with the forward Euler method, where:

$$\boldsymbol{\theta}(t + \delta t) \approx \boldsymbol{\theta}(t) + A^{-1}(t)C(t)\delta t$$

Usually, the matrix $A(t)$ is not well-conditioned, so instead, we solve for each t:

$$\underset{\theta(t) \in \mathbf{R}^n}{\arg\min} \, \|A(t)\boldsymbol{\theta}(t)' - C(t)\|$$

To carry this entire procedure out, the first step is to discretize $\widetilde{\mathfrak{G}}$ with periodic boundary conditions, and then decompose $\widetilde{\mathfrak{G}}$ into unitary operations $\sum \lambda_i h_i$ that can be represented on a quantum circuit.

Consider a 16 by 16 grid of qubit states, achieved by using 8 qubits, which has the following states $\{|0\rangle, |1\rangle, \cdots |2^8 - 1\rangle\}$. We embed the solution u_{ij} into this grid, where the value of u_{ji} corresponds to the probability amplitude of the qubit state it is located at.

To discretize the generator $\widetilde{\mathfrak{G}}$, we use the stencil method, where we use the effect of $\widetilde{\mathfrak{G}}$ on the qubit grid to represent it:

$$\widetilde{\mathfrak{G}} = \sum_{i_1=0}^{2^4-1} \sum_{i_2=0}^{2^4-1} \sum_{j_1=0}^{2^4-1} \sum_{j_2=0}^{2^4-1} [\widetilde{\mathfrak{G}}]_{(i_1,i_2)\times(j_1,j_2)} |i_1, i_2\rangle \langle j_1, j_2|$$

First, express the first-order derivative operators via the following approximation:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j} - u_{ij}}{\Delta x}$$
$$\frac{\partial u}{\partial x} \approx \frac{u_{i,j+1} - u_{ij}}{\Delta y}$$

For second-order derivative operators, we have

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2}$$
$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y}$$

Then, $\widetilde{\mathfrak{G}}$ can be discretised as the following:

$$[\widetilde{\mathfrak{G}}]_{(i_1,i_2)\times(j_1,j_2)} = \begin{cases} \dfrac{x^4y^2}{\Delta x^2} + \dfrac{x^2y^3+y^4}{\Delta y^2} - 1 & \text{if } (i_1,i_2) = (j_1,j_2) \\ \dfrac{1}{2}\dfrac{x^4y^2}{\Delta x^2} \mp \dfrac{1}{2}\dfrac{x}{\Delta x} & \text{if } (i_1,i_2) = (j_1 \pm 1, j_2) \\ \dfrac{1}{2}\dfrac{x^2y^3+y^4}{\Delta y^2} \mp \dfrac{1}{2}\dfrac{y}{\Delta y} & \text{if } (i_1,i_2) = (j_1, j_2 \pm 1) \\ \dfrac{1}{4}\dfrac{x^3y^{5/2}}{\Delta x \Delta y} & \text{if } (i_1,i_2) = (j_1 \pm 1, j_2 \pm 1) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The next step is to use this discretization to express $\widetilde{\mathfrak{G}}$ as linear combination of Unitary operators.

Notice that due to the stencil method, $\widetilde{\mathfrak{G}}$ only attains non zero value along some diagonal and its offset by 1. Hence, we write it in the following form:

$$\begin{aligned}
\widetilde{\mathfrak{G}} = &\sum_{j_1=0}^{2^4-1}\sum_{j_2=0}^{2^4-1}\left(\frac{x^4y^2}{\Delta x^2}+\frac{x^2y^3+y^4}{\Delta y^2}-1\right)|j_1,j_2\rangle\langle j_1,j_2| \\
&\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2}-\frac{1}{2}\frac{x}{\Delta x}\right)|j_1+1,j_2\rangle\langle j_1,j_2| \\
&\sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2}+\frac{1}{2}\frac{x}{\Delta x}\right)|j_1-1,j_2\rangle\langle j_1,j_2| \\
&\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2}-\frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2+1\rangle\langle j_1,j_2| \\
&\sum_{j_1=1}^{2^4-1}\sum_{j_2=1}^{2^4-1}\left(\frac{1}{2}\frac{x^2y^3+y^4}{\Delta y^2}+\frac{1}{2}\frac{y}{\Delta y}\right)|j_1,j_2-1\rangle\langle j_1,j_2| \\
&\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{4}\frac{x^3y^{5/2}}{\Delta x\Delta y}\right)|j_1\pm 1,j_2\pm 1\rangle\langle j_1,j_2|
\end{aligned}$$

After the periodic boundary condition $|2^n\rangle = |0\rangle$, $|-1\rangle = |2^n-1\rangle$, and x,y are x_{j_1}, y_{j_2} respectively.

First define the following Operators, where

$$\begin{aligned}
V_+(n) &= \sum_{i=0}^{2^n-2}|i+1\rangle\langle i| = \mathrm{CycInc}(n)\frac{1}{2}(C^nZ+I^{\otimes n}) \\
V_-(n) &= \sum_{i=1}^{2^n-1}|i-1\rangle\langle i| = \frac{1}{2}(C^nZ+I^{\otimes n}\mathrm{CycDec}(n) \\
D(n) &= \sum_{i=0}^{2^n-1}i\,|i\rangle\langle i| = \frac{2^n-1}{2}I^{\otimes n}-\sum_{i=1}^{n}2^{n-i-1}Z_i
\end{aligned}$$

Where C^nZ is a n-qubit controlled pauli Z gate, I is the identity gate, and Z_i is the pauli Z gate acting on the i-th qubit.

The following

$$\begin{aligned}
\mathrm{CycInc}(n) &= \sum_{i=0}^{2^n-1}|i+1\rangle\langle i| \\
\mathrm{CycDecc}(n) &= \sum_{i=}^{2^n-1}|i-1\rangle\langle i|
\end{aligned}$$

are Cyclic Increment and Cyclic Decrement gates respectively.

With these unitary operations, we continue to build up the components that allows us to express $\widetilde{\mathfrak{G}}$:

$$\begin{aligned}
V_\pm^{(k)}(n) &= I^{\otimes k-1}\otimes V_\pm(n)\otimes I^{\otimes 2-k} \quad \text{k} = 1,\ 2 \\
D^{(k)}(n) &= I^{\otimes k-1}\otimes D(n)\otimes I^{\otimes 2-k} \quad \text{k} = 1,\ 2
\end{aligned}$$

The following are examples of what these operators can be used to express in conjunction:

$$V_+^{(1)}(4)[D^{(1)}(4)]^m = \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2} j_1^m \left|j_1+1, j_2\right\rangle \left\langle j_1, j_2\right|$$
$$V_-^{(2)}(4)[D^{(2)}(4)]^m = \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2} j_2^m \left|j_1, j_2-1\right\rangle \left\langle j_1, j_2\right|$$

The purpose of the power m is to express $\widetilde{\mathfrak{G}}$ when we approximate the functions inside using taylor expansion at $(x, y) = (0, 0)$. For example, using second-order Taylor expansion, we have

$$\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} - \frac{1}{2}\frac{x}{\Delta x}\right) = -\frac{1}{2\Delta x}x \approx -\frac{1}{2\Delta x}\cdot \Delta x j_1 \quad \forall (x, y) = (x_{j_1}, y_{j_2})$$

Then this term in $\widetilde{\mathfrak{G}}$ can be expressed as

$$\sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2}\left(\frac{1}{2}\frac{x^4y^2}{\Delta x^2} - \frac{1}{2}\frac{x}{\Delta x}\right)\left|j_1+1, j_2\right\rangle \left\langle j_1, j_2\right| \approx \sum_{j_1=0}^{2^4-2}\sum_{j_2=0}^{2^4-2} -\frac{1}{2\Delta x}\cdot \Delta x j_1 \left|j_1+1, j_2\right\rangle \left\langle j_1, j_2\right| = -\frac{1}{2}V_+^{(1)}(4)[D^{(1)}(4)]^m$$

We apply this technique to every term to express $\widetilde{\mathfrak{G}}$ in terms of unitary operators.
Difficulty I am encountering:

- Running this on a quantum circuit, I ran into some software issues where the qiskit package couldn't let me run the predefined VarQITE algorithm
- Turn unitary operation into actual gates on a circuit. For some unitary operation such as CycInc(n), not sure how to apply on a circuit